Multi-period Efficient Frontiers and Sharpe Ratios under the Buy and Hold and the Rebalancing Strategies

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Abstract

We analyze and compare multi-period efficient frontiers and Sharpe Ratios under Buy and Hold and Rebalancing strategies. We show that the single-period capital market line preserves its linearity in the multi-period case under Buy and Hold, whereas it becomes a concave curve under Rebalancing. This finding makes it clear that benchmarks used as measures of performance could likely be inappropriate if the underlying theory is not accounted for in their selection. Further, we show that the multi-period efficient frontier is a hybrid of the Rebalancing frontier for riskless lending/risky equity combinations and of the Buy and Hold frontier for riskless borrowing/risky equity combinations. This finding suggests that more risk tolerant investors can justify their selection of the Buy and Hold Strategy for multi-period investing. Moreover, we show that multi-period Sharpe Ratios are unaffected by the riskiness of the portfolio under Buy and Hold, but inversely related to portfolio risk under Rebalancing. Our analysis indicates that an understanding of the form of the multi-period efficient frontier is important to portfolio performance measurement.

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1. Introduction

1.1 Investment Horizon

The efficiency of capital markets and the persistent popularity of the capital asset pricing model (CAPM), mainly among practitioners, have resulted in the creation of index funds as proxies for the theoretical market portfolio and the investor focus on combinations of short-term money market instruments and stocks. The traditional CAPM assumes that all investors have the same single-period investment horizon. In reality, however, individual investors have different temporal consumption patterns and hence heterogeneous investment horizons. Tobin (1965) was the first to analyze the effect of the heterogeneous investment horizons on portfolio choice. The classical Markowitz (1959) model of portfolio analysis uses a single period, mean-variance approach to define an optimal portfolio. However, when investors have heterogeneous horizons, a multi-period strategy must be adopted.

The two analytically tractable competing multi-period strategies are Buy-and-Hold and Rebalancing. There is abundant evidence in the financial literature that for long term investment horizons the Rebalancing strategy outperforms the Buy-and-Hold strategy. Evans (1968, p. 329) claims that, for portfolios of assets, the returns of the two strategies will tend to differ over time, with the Rebalancing strategy yielding risk-adjusted returns superior to the Buy-and-Hold strategy if, (1) asset values fluctuate randomly around the intrinsic value and, (2) the market is characterized by an underlying upward trend over the long run. Empirical evidence by Evans (1968), Blume and Friend (1974), and more recently Arnott and Lovell (1990), Buetow et al. (2002) and Plaxco and Arnott (2002) all suggest that the Rebalancing strategy can yield returns which are superior to those yielded by the Buy-and-Hold strategy.[1] Further, Evans (1968, pp. 337, 341) states that "it appears highly reasonable to assert that one should replace the Buy-and-Hold strategy with the Rebalancing strategy as the appropriate standard against which to measure the performance of mutual fund portfolios and portfolios in general." Similarly, Buetow et al. (2002), using actual and simulated data, find that more frequent periodic rebalancing is better than less frequent.[2] Levy and Samuelson (1992) show that under Rebalancing the CAPM holds with diverse holding periods when, say, utility functions are quadratic, one-period distributions are normal and terminal wealth is lognormally distributed. Plaxco and Arnnott (2202, p. 21) write that, "A buy-and-hold strategy leads to a drifting portfolio mix that is both difficult to justify and unprofitable when compared to the appropriate [rebalancing] alternative," and that, "The alternative of allowing the mix to drift without limit is neither predictable nor prudent. The essential therefore is not *whether* to rebalance, but instead *how* to rebalance [their emphasis]." These studies suggest that for unleveraged portfolios, the Rebalancing strategy will perform better than the Buy-and-Hold strategy. Furthermore, the longer the investment horizon, the more pronounced the difference would be. Would these findings also hold true for leveraged portfolios?

In this paper we show, with the help of a numerical illustration, that in the multi-period case under Buy-and-Hold, the capital market line preserves its linearity, while under Rebalancing it becomes a concave curve. Our example further illustrates that for riskless lending/risky equity portfolios, the efficient frontier is the Rebalancing curve, whereas for combinations of riskless borrowing/risky equity (a leveraged portfolio) the linear Buy-and-Hold frontier dominates. The result is a kinked efficient frontier that is a hybrid of the two strategies — in the unleveraged portion of the efficient frontier the Rebalancing strategy dominates the Buy-and-Hold strategy, whereas the Buy-and-Hold strategy dominates the Rebalancing strategy for leveraged portfolios. This finding is in agreement with Perold and Sharpe's (1988) suggestion that the investor's risk

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[1] One of the conclusions reached by Blume and Friend (1974, p. 259) reads as follows:

> Although, over a long period, average five-year rates of return realized by rebalancing tended to be only moderately higher than those for buy-and-hold strategies, the differences in any five-year period, while possibly not statistically significant at the five percent level, are frequently of sufficient size to be of major importance to an investor.

[2] Since the continuous-time economy is a special, limit case of the discrete-time economy, one would expect the RB strategy to obtain there too. Indeed, Merton (1971) shows that when asset prices are lognormally distributed and the investor has a power utility function (or more general utility function with constant relative risk aversion), the optimal portfolio strategy is to continuously rebalance the portfolio to a constant mix between the risky stocks and the riskless bond.

tolerance will determine which strategy is appropriate and, more specifically, what particular risk/return configuration is the appropriate choice.[3]

1.2 Risk/Return Measures

Tobin (1965) was the first to develop a relationship between the risk and return measures of the single-period investment horizon and those of the multi-period investment horizons. Levy (1972) furthered the development by using the Sharpe Ratio, or the reward to variability index, to analyze the effect of the investment horizon on this measure of performance. Levy (1972) and Hodges, Taylor, and Yoder (1997) show that the Sharpe Ratio of an investment changes with the investment horizon. Further, Hodges, Taylor, and Yoder (1997) show that the lengthening of the investment horizon leads to a higher multi-period Sharpe Ratio for low-risk investments than for high-risk investments. The illustration we develop in this paper indicates that the Sharpe Ratio is unaffected by the riskiness of the portfolio under Buy-and-Hold and is inversely related to riskiness under Rebalancing, which is in agreement with the Hodges, Taylor, and Yoder (1997) finding.

This work further extends this literature by 1) providing a theoretical basis for the previous empirical results, 2) showing the importance of ascertaining an investor's risk tolerance before prescribing a long term strategy, and 3) interpreting three relatively recent articles in light of our conclusions to highlight the practical application of our findings.

The remainder of this paper is organized as follows. In section 2 we derive multi-period Buy-and-Hold and Rebalancing formulas to produce multi-period efficient frontiers even though the Buy-and-Hold multi-period efficient frontiers have been empirically shown to be dominated by homologous Rebalancing frontiers.[4] In section 3, we show that while the Rebalancing efficient frontier is superior for combinations of the risky asset and the riskless asset without borrowing, the Buy-and-Hold efficient frontier becomes dominant for leveraged portfolios. In section 4, we show that the multi-period Sharpe Ratio decreases with the riskiness of the portfolio under Rebalancing but is unaffected by changes in the riskiness of the portfolio under Buy-and-Hold. In section 5, we apply the insights gained in this paper to point out conceptual shortcomings in three recent contributions to this area. Finally, section 6 summarizes our findings and offers concluding comments.

2. Multi-period Risk-Return Comparisons under Buy-and-Hold and Rebalancing

Assume that the investment horizon contains n equal periods. The Buy-and-Hold (BH) strategy assumes that the original selection is held with no rebalancing for the entire n-period horizon. Unlike the BH strategy the Rebalancing (RB) strategy assumes the investment proportion between risky and riskless assets is rebalanced so as to maintain the same initial weight each period. Following Tobin (1965), assume that returns are serially independent and stationary and

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[3] Perold and Sharpe (1988) consider two additional strategies, the constant proportion portfolio insurance and the option-based portfolio insurance. However, their analysis, unlike ours, is not performed in discrete time and is not static.

[4] See Evans (1968), Blume and Friend (1974), and Buetow (2002).

that the investor is considering a portfolio of two assets — a stock index and a constant return riskless asset. For this portfolio the single period return $R_p(1)$, is:

$$R_p(1) = \alpha R_m(1) + (1-\alpha)R_f(1) \tag{1}$$

where: $R_m(1)$ = the single period rate of return of the risky asset with expected return $E(R_m(1))$ and standard deviation of $\sigma_m(1)$,
$R_f(1)$ = the single period rate of return of the riskless asset with expected return of $E(R_f(1)) = R_f(1)$ and standard deviation $\sigma_f(1)=0$,
α = the proportion of the portfolio invested in the risky asset and $(1-\alpha)$ the proportion invested in the riskless asset.

The single period expected portfolio return $E(R_p(1))$ is:

$$E(R_p(1)) = \alpha E(R_m(1)) + (1-\alpha)R_f(1) \tag{2}$$

and the single-period portfolio variance $\sigma^2_{p(1)}$ is:

$$\sigma^2_{p(1)} = \alpha^2\sigma^2_{m(1)} \tag{3}$$

Assuming $E(R_m(1))>R_f(1)$, equations (2) and (3) show that the higher the proportion of the money invested in the all equity index, the higher the expected portfolio return and the higher the variance of the portfolio return.

2.1 The Buy and Hold Formulas[5]

Under the BH strategy the formula for the n-period portfolio expected rate of return is:

$$E(R_p(n)) = \alpha\left[(1+E(R_m(1)))^n - 1\right] + (1-\alpha)\left[(1+R_f(1))^n - 1\right] \tag{4}$$

and, the formula for the n-period portfolio variance is:

$$\sigma_p^2(n) = \alpha^2\left\{\left[(1+E(R_m(1)))^2 + \sigma_m^2(1)\right]^n - \left[1+E(R_m(1))\right]^{2n}\right\} \tag{5}$$

Equations (4) and (5) show how the multi-period return, risk, and proportion parameters under BH are related to the single-period return and risk parameters in equations (2) and (3).

[5] For the derivation of the expected return and variance formulas for both the buy and hold and rebalancing strategies, please see Appendix A.

2.2 The Rebalancing Formulas

Under the RB strategy, the n-period portfolio expected rate of return is:

$$\begin{aligned} E(R_p(n)) &= \left[1 + E(R_p(1))\right]^n - 1 \\ &= \left[1 + \left(\alpha E(R_m(1)) + (1-\alpha)R_f(1)\right)\right]^n - 1 \end{aligned} \tag{6}$$

and the associated n-period portfolio variance is:

$$\begin{aligned} \sigma_p^2(n) &= \left[\left(1 + E(R_p(1))\right)^2 + \sigma_p^2(1)\right]^n - \left[1 + E(R_p(1))\right]^{2n} \\ &= \left[\left(1 + \alpha E\left(R_m(1)\right) + (1-\alpha)R_f(1)\right)^2 + \alpha^2\sigma_{m(1)}^2\right]^n - \left[1 + \alpha E\left(R_m(1)\right) + (1-\alpha)R_f(1)\right]^{2n} \end{aligned} \tag{7}$$

Rebalancing implies periodic adjustment of the portfolio to re-establish the weights originally allocated to each asset. For our specific purposes the rebalancing strategy means rebalancing to a portfolio with constant mix between the risky stock and the riskless bond. When the original weighting proportions require either $\alpha = 1$ or $\alpha = 0$, then the expected return and variance from the BH strategy will equal the expected return and variance from the RB model. In other words, for individual assets the performance of the two strategies is identical.

3. Comparing Multi-period Efficient Frontiers under BH and RB Strategies

We know that in the multi-period case different strategies have different risk/return configurations and thus lead to different consequences. To emphasize fundamentals we, along with Arnott and Lovell (1990), Perold and Sharpe (1988), and Van Eaton and Conover (1998), focus on a choice between only two assets — stocks and riskless bonds.

Now comparison of equations (2) and (3) with (4) and (5) shows that they are functionally identical. Consequently, the one-period Capital Market Line (CML) preserves its linearity in the multi-period case under BH. However, in the case of the multi-period efficient allocation under RB, the resulting optimal mean-standard deviation allocations will likely be curvilinear/concave and superior to the BH attainable allocations in the range from zero to 100 percent invested in the risky asset.[6]

To illustrate and contrast the different solutions provided by the two strategies, we assume the same distributional parameters for the risk-free and risky assets and the same investment horizons as Van Eaton and Conover (1998).[7] Further, to clearly and dramatically contrast the BH efficient frontier to the RB efficient frontier, we focus our analysis on n=30 period

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[6] The empirical literature shows that for $0<\alpha<1$ the risk/return configuration of the rebalancing strategy is above that of the buy and hold strategy. For $\alpha=0$ and $\alpha=1$ the two strategies are identical. It follows that the rebalancing strategy must be concave and above the buy and hold strategy.

[7] The single-period parameter values used to calibrate our two models of multi-period efficient frontiers reflect the historical experience of the U.S. financial markets.

investment horizon. Table I (Table II) provides the means and standard deviations for different risky-asset allocations under the BH (RB) strategy and for n=1, 2, 5, 7, 10, 20 and 30 years.[8]

Inspection of Table I and Table II suggests the following trade-offs between expected return and risk for the two strategies:

1. For every $n \geq 2$ investment horizon and original equity weight $\alpha < 1$, BH offers the higher expected return but RB has the attraction of lower risk.
2. Conversely, for every $n \geq 2$ and $\alpha > 1$, the RB strategy offers the higher expected return but the BH strategy has the attraction of lower risk.

Thus comparing risk/return characteristics for a given α will not provide uniform preferences. Only the comparison of the BH efficient frontier to the RB efficient frontier will provide the correct selection within the mean-standard deviation framework.

Figure 1 (Figure 2) shows the corresponding mean-standard deviation efficient frontiers under the BH (RB) strategy. Inspection of these figures shows that, as expected, the BH efficient frontiers are indeed straight lines for all investment horizons whereas the RB efficient frontiers are curvilinear/concave for all $n \geq 2$.

In Figure 3 the juxtaposition of the RB efficient frontier on the homologous BH efficient frontier shows that for a given multi-period risk $\sigma_p(n)$ the RB expected return is higher than the corresponding BH expected return when $\alpha < 1$. However, for $\alpha > 1$ the same $\sigma_p(n)$ provides a higher BH expected return than the corresponding RB expected return. Figure 3 was constructed using the means and standard deviations for different risky-asset allocations (α=0.00, 0.05, 0.10, …, 1.45, 1.50) under both the BH and RB strategies when n=30. It should be noted that the same $\sigma_p(n)$ does not have the same α under BH and RB. The risk return configurations under both BH and RB are identical only for $\alpha=0$ and $\alpha=1$. The implications elicited from observing Figure 3 is that the Capital Market Line when n=1 turns into a Capital Market Curve that is a hybrid of the RB frontier for $0 \leq \alpha \leq 1$ and of the BH frontier for $\alpha > 1$ when $n \geq 2$.

4. The Multi-period Sharpe Ratio, SR(n)

When the investor chooses the best portfolio of stocks in the presence of a riskless asset, the chosen risky portfolio m has the highest expected return for a given level of risk. If borrowing and lending rates are the same then the efficient frontier is a straight line, the Capital Market Line. How much to invest in portfolio m, the risky asset, depends on the investor's level of risk aversion. But regardless of the investor's attitude towards risk the composition of stocks in portfolio m is unchanged. Thus the best stock portfolio m offers the highest possible ratio of expected reward to risk, or the best Sharpe Ratio, SR.[9] Formally, the single-period Sharpe Ratio is defined as

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[8] In this paper mean is defined as mean return or mean wealth relative $E(1+R_p)$, not as mean rate of return $E(R_p)$.

[9] It is naïve to assume that in the absence of specific information about 1) strategy, 2) investment horizon, and 3) the presence or absence of leverage, the highest SR corresponds to the most efficient portfolio.

$$SR_{(1)} = \frac{E\left(R_p(1)\right) - R_f(1)}{\sigma_p(1)} \tag{8}$$

In the multi-period case the Sharpe Ratio is defined as

$$SR_{(n)} = \frac{E\left(R_p(n)\right) - \left[(1 + R_f(1))^n - 1\right]}{\sigma_p(n)} \tag{9}$$

$SR_{(n)}$ is a performance index that measures the risky portfolio's multi-period excess return, $E\left(R_p(n)\right) - \left[(1 + R_f(1))^n - 1\right]$, per unit of multi-period risk, $\sigma_p(n)$. Equation (9), in light of the expected returns and standard deviations developed in equations (4) and (5) for the BH strategy and equations (6) and (7) for the RB strategy, shows that the $SR_{(n)}$ is a complex, nonlinear function of $E(R_m(1))$, $R_f(1)$, α, $\sigma_m^2(1)$, n, and the chosen strategy.

In empirical work, estimates of the above parameters are typically used with $SR_{(1)}$, i.e., the actual observed average rate of return, the actual average interest rate, and the standard deviation of the actual returns. Ex post, the higher the $SR_{(1)}$,the more successful the portfolio's management when compared with other funds or the aggregate market. In our case we use the $SR_{(1)}$ in an expectational, ex ante sense. It is said that the SR value is meaningless without relating it to a standard. In this paper, we analyze the behavior of the standard multi-period SR under the RB and BH strategies.

Table III reports the Sharpe Ratio as a function of the investment horizon, the chosen multi-period strategy (BH or RB), and the equity weight, α. Applying equation (9) to the data from Table I, we obtain the reported $SR_{(n)}$ values. As shown in Panel A, with a BH strategy $SR_{(n)}$ is not a function of equity weight, α. The fact that $SR_{(n)}$ is the same regardless of α is a feature of the straight line multi-period efficient frontier under BH. $SR_{(n)}$,however, increases with n up to a point and then decreases.

As Panel B in Table III shows, and Figure 4 illustrates, for $n \geq 2$ $SR_{(n)}$ is a function of equity weight, α, under RB. The higher α, or the riskier the portfolio, the smaller the SR value for a given investment horizon $n \geq 2$. Further, for a given $\alpha < 1$ it appears that $SR_{(n)}$ increases with n. On the other hand, given $\alpha \geq 1$ it appears that $SR_{(n)}$ increases with n up to a point and decreases afterwards. This $SR_{(n)}$ behavior is a direct result of the concavity of the multi-period efficient frontier under RB.

The results in Table III and Figure 4 are in general agreement with the empirical performance rankings obtained by Hodges, Taylor, and Yoder (1997) who analyze the performance of traditional asset classes. They show that under RB, relatively low-risk long-term corporate bonds have Sharpe Ratios that increase with the length of the investor's horizon and that very risky small cap stocks have Sharpe Ratios that, relative to common stocks, decrease the most for very long horizons, say n=30 years.

5. Critical Review of Recent Contributions in the Literature

Now that we have analyzed the shape of the multi-period efficient frontier and the behavior of the Sharpe Ratio under BH and RB, we apply the gained insights to analyze three relatively recent contributions in this area of finance.

5.1 The Albrecht paper.

Albrecht (1998) uses the Sharpe Ratio under RB as a measure of performance. He rejects the view reflected in Table III, Panel B and Figure 4, that "low-risk investments grow more attractive at longer investment horizons as their Sharpe Ratios improve relative to high-risk investments." (p. 44). He asserts that the long-term standard deviation of returns can be misleading as an indicator of risk. Albrecht considers two investments, a low risk investment A (with $\alpha_A<1$) and a high-risk investment B (with $\alpha_B>1$). Figure 5 (based on Albrecht's Figure 1) shows that both investments have the same single-period Sharpe Ratio and thus a combination of B with lending will obtain risk/return characteristics identical to A. However, as Figure 6 (based on Albrecht's Figure 2) "shows," when the investment horizon is n=10 years, the Sharpe Ratio of the riskier investment B (measured by the slope of the straight line $D_{10}B$) is lower than that of the less risky investment A (measured by the slope of the straight line $D_{10}A$). Whereas under RB all combinations of A or B with the risk-free asset will lie on the concave efficient frontier D_{10} A B C, Albrecht incorrectly considers the straight line that passes through D_{10} and A(B) as the locus of all combinations of investment A(B) and the risk-free security that have the same Sharpe Ratio to conclude that "an investor could use a combination of investment A and a loan to achieve a higher return at the same standard deviation as investment B [reflected in A′]. Alternatively, the investor could combine investment B with a risk-free asset to duplicate the standard deviation of investment A, but at a lower return [reflected in B′](p.44). Albrecht has failed to take into account the concavity of the RB efficient frontier that causes points A and B to lie on the same curve. This leads him to erroneously conclude that the multi-period standard deviation of returns can be an inadequate and misleading indicator of risk.

5.2 The Marshall Paper.

Marshall (1994) examines the role of the investor's horizon on the choice of optimal portfolios. He concludes that, under RB, investors should choose progressively less risky single-period portfolios as their investment horizons grow shorter, even if they do not become more risk averse. When a constant-return riskless asset is available and stationarity and independence of successive portfolio returns are assumed, he obtains a tangency portfolio that maximizes the Sharpe Ratio that is different for each horizon.

In the presence of a riskless asset the efficient frontier is a straight line when n=1 and a concave curve when n≥2. We believe Marshall's formulation of the multi-period efficient frontier under RB is incorrect since his tangency portfolios are distinctive and horizon-specific. Our analysis assumes that, under RB and single-period efficient set stationarity, there is <u>no</u> multi-period-specific tangency portfolio of all risky assets that is different from the single-period tangency portfolio. The error of Marshall's analysis is that he used a BH-consistent straight line passing through the multi-period riskless rate to select the multi-period tangent portfolio. However, as

Levy and Samuelson (1992) show, under RB the CAPM holds with diverse holding periods when, say, either utility functions are quadratic, or one-period distributions are normal or terminal wealth is lognormally distributed. Since Marshall assumes lognormal returns, theorem 8 in Levy and Samuelson (1992, p. 1537) is pertinent.

> It says (with symbols changed to reflect our nomenclature),
> assume that the n-period returns are lognormally distributed, the portfolio returns are independent over time, and that investors are allowed to revise their investment portfolios n-1 times. Then the two fund Separation Theorem holds and the Sharpe-Lintner CAPM is intact even with diverse holding periods.

In other words, all investors independent of their holding period, n, will choose some mix of portfolio m and the riskless asset. We conclude that whether the efficient frontier reflects the investor's personal security universe or reflects the CAPM assumptions, the tangent portfolio m is the same regardless of the holding period.

Marshall's paper has contributed to the portfolio selection literature by advocating a choice criterion that is suited to a multiperiod environment and is consistent with the traditional utility approach. However, his claim that investors should shift to less risky equity portfolios as their horizons shorten, does not agree with the correct analysis of the multi-period portfolio selection under RB.

5.3 The Van Eaton and Conover Paper.

Van Eaton and Conover (1998) contributed to the debate about whether rational investors, using a BH strategy, prefer larger or smaller equity percentages in their portfolios over longer investment horizons. Using their data, we have demonstrated that the multi-period RB efficient frontier dominates the BH efficient frontier for combinations of the risky asset and riskless asset without borrowing. However, as Figure 3 shows, for leveraged portfolios the BH efficient frontier becomes dominant. Although we believe that the majority of investors select lending-risky asset combinations, investors with moderate or aggressive risk tolerance and long-investment horizons will select borrowing-risky asset combinations and analysis by Siegel (1999) bears out this belief.

Van Eaton and Conover (1998) did not heed Perold and Sharpe's (1988) suggestion that the investor's risk tolerance will determine which strategy is appropriate and, more specifically, what particular risk/return configuration is the appropriate choice. As our previous comments and Figure 3 suggest, investors with high risk aversion reflected in iso-utility I_1, will choose from the RB portion of the capital market curve (see Figure 3, I_1) whereas investors with low risk aversion will choose from its BH portion (see Figure 3, iso-utility I_3).[10] Van Eaton and Conover, however, first assume a strategy, the BH strategy, and then allow the investor to make the risk/return selection from the n-period BH efficient frontier.

[10] It is also possible for the investor to have an iso-utility curve that is tangent to both the BH and the RB portion of the capital market curve (see Figure 3, I_2).

Since simultaneous consideration of competing strategies such as BH and RB can result in higher expected utility levels for investors and a hybrid capital market curve, the a priori adoption by Van Eaton and Conover of the BH framework will generally produce second-best utility results. In analyzing the relation between optimal equity allocation and investment horizon, they used the following general mean-variance expected utility function:

$$U(n) = E(R_p(n)) - \frac{A}{2}(\sigma_p(n))^{\beta} \quad (10)$$

where:

$U(n)$ = investor's expected utility from portfolio p, invested for n periods,

A = risk-aversion (Slope) parameter, and

β = the exponent on standard deviation that determines the curvature of the utility isoquant in mean-standard deviation space ($1<\beta<\infty$).

Markowitz (1991) reviews various mean-variance approximations to expected utility functions. Let $f(E(R_p), \sigma_p^2)$ be the approximation, $U(R)$ be the exact utility function, and $U''(R)$ be its second derivative with respect to R. He finds that, almost without exception, the following Taylor-based approximation (with symbols changed to reflect our nomenclature),

$$f(E(R_p), \sigma_p^2) = U(E(R_p)) + 0.5U''(E(R_p))\sigma_p^2 \quad (10')$$

provides the best fit to the actual utility function under consideration. Comparing Markowitz's (1991, p. 473, eq. (2)) equation (10′) to equation (10) shows that, in general, they are incompatible and thus any conclusions derived from Van Eaton and Conover's flexible, but arbitrary, expected utility function should be viewed with suspicion.

Thus,we believe Van Eaton and Conover err on two levels. First they neglect to consider both strategies simultaneously and second, their utility function is so flexible that it accommodates any type of result. Most expected utility functions would not take such a fluid form.

6. Summary and Concluding Comments

In this paper we use two-asset class portfolios under Rebalancing and under Buy-and-Hold strategies and, based on numerical analysis of the theoretical formulas we derived, we find that Rebalancing improves the multi-period performance of Buy-and-Hold for unleveraged portfolios. However, for leveraged portfolios the reverse obtains. The Buy-and-Hold strategy has a long pedigree and its performance is used as a yardstick against which other strategies or trading rules are compared. For example, Roll (1994, p. 71), scholar and practicing money manager, commenting on the practical exploitation of market inefficiencies discovered by finance researchers, provides the following cautionary statement: "… Many of these effects are surprisingly strong in the reported empirical work, but I have never found one that worked in practice, in the sense that it returned more than a buy-and-hold strategy." Even though Roll's comments are about active stock selection strategies and not the comparison between rebalancing and the buy and hold strategy, we believe his argument would be strengthened by considering the rebalancing strategy as the appropriate yardstick. Indeed Roll's view, however, has been recently challenged, both theoretically and empirically. The abundance of literature cited in this paper is evidence of this new preference for the rebalancing strategy.

Our work clarifies precisely under what circumstances the Buy-and-Hold strategy should be used as the standard and under what circumstances the Rebalancing strategy should be preferred. As Figure 3 shows, since the multi-period efficient frontier has a kink at $\alpha=1$, it is not uniformly concave. We then expect that for α values around 1, the investor might be indifferent between Rebalancing and Buy-and-Hold strategies, but for smaller α values the unleveraged investor will adopt the Rebalancing strategy as suggested by the more recent literature and for higher α values the leveraged investor will adopt the Buy-and-Hold strategy, a more traditional approach. Levy and Samuelson (1992, p. 1529) state that "without portfolio revisions the CAPM does not follow even with quadratic utility functions." This means that when $\alpha>1$ the multi-period investor will follow the Buy-and-Hold strategy and the CAPM will be invalid! In agreement with Levy and Samuelson, our findings highlight the need for investors to first establish a risk tolerance before determining a strategy to follow.

In evaluating portfolio performance in mean-standard deviation space, the Sharpe Ratio, SR(n), is a popular metric. As we show, however, the multi-period SR(n), $n\geq2$, does not possess the simplicity and clarity of interpretation of the single-period SR(1). As the numerical illustrations of our mathematical formulas show and the simulations by Hodges, Taylor, and Yoder (1997) confirm, the SR(n) under Rebalancing should be interpreted with care. Under Buy-and-Hold, since SR(n) is unaffected by the proportion of the risky asset in the portfolio, α, its interpretation in empirical studies should be more straightforward. This finding highlights the need for caution in metric selection when measuring the performance of portfolios.

Finally, in light of the insights gained in this paper, we critically comment on two recent contributions in the Financial Analysts Journal and a third contribution in the Financial Review. We suggest that Albrecht (1998) errs in concluding that the multi-period standard deviation of returns can be an inadequate and misleading indicator of risk. His conclusion ignores the concavity of the Rebalancing efficient frontier. Marshall (1994) errs in suggesting that investors should lower their risk tolerance as investment horizons shorten. His conclusion once again ignores the concavity of the Rebalancing efficient frontier. Finally, Van Eaton and Conover (1998) err by not considering both competing strategies simultaneously. The question they pose must first consider the investor's risk tolerance before the appropriate strategy is selected. Consideration of both strategies would have improved the quality of their analysis.

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Table I
Means and standard deviations for various equity allocations under the buy and hold (BH) strategy: Horizons n = 1, 2, 5, 7, 10, 20, 30 years

Equity Weight	n = 1		n= 2		n = 5		n = 7		n = 10		n = 20		n = 30	
	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev
0.00000	1.04000	0.00000	1.08160	0.00000	1.21665	0.00000	1.31593	0.00000	1.48024	0.00000	2.19112	0.00000	3.24340	0.00000
0.05000	1.04200	0.01000	1.08584	0.01540	1.22929	0.03148	1.33583	0.04421	1.51418	0.06836	2.31461	0.22867	3.58436	0.66556
0.10000	1.04400	0.02000	1.09008	0.03081	1.24192	0.06297	1.35572	0.08843	1.54811	0.13671	2.43811	0.45734	3.92532	1.33112
0.15000	1.04600	0.03000	1.09432	0.04621	1.25455	0.09445	1.37562	0.13264	1.58205	0.20507	2.56160	0.68602	4.26629	1.99668
0.20000	1.04800	0.04000	1.09856	0.06162	1.26719	0.12593	1.39551	0.17685	1.61598	0.27343	2.68509	0.91469	4.60725	2.66224
0.25000	1.05000	0.05000	1.10280	0.07702	1.27982	0.15741	1.41540	0.22107	1.64991	0.34178	2.80858	1.14336	4.94821	3.32780
0.30000	1.05200	0.06000	1.10704	0.09242	1.29246	0.18890	1.43530	0.26528	1.68385	0.41014	2.93207	1.37203	5.28918	3.99336
0.35000	1.05400	0.07000	1.11128	0.10783	1.30509	0.22038	1.45519	0.30949	1.71778	0.47850	3.05557	1.60071	5.63014	4.65892
0.40000	1.05600	0.08000	1.11552	0.12323	1.31772	0.25186	1.47509	0.35370	1.75172	0.54685	3.17906	1.82938	5.97110	5.32448
0.45000	1.05800	0.09000	1.11976	0.13864	1.33036	0.28334	1.49498	0.39792	1.78565	0.61521	3.30255	2.05805	6.31206	5.99004
0.50000	1.06000	0.10000	1.12400	0.15404	1.34299	0.31483	1.51488	0.44213	1.81958	0.68356	3.42604	2.28672	6.65303	6.65560
0.55000	1.06200	0.11000	1.12824	0.16944	1.35562	0.34631	1.53477	0.48634	1.85352	0.75192	3.54953	2.51539	6.99399	7.32116
0.60000	1.06400	0.12000	1.13248	0.18485	1.36826	0.37779	1.55467	0.53056	1.88745	0.82028	3.67302	2.74407	7.33495	7.98672
0.65000	1.06600	0.13000	1.13672	0.20025	1.38089	0.40927	1.57456	0.57477	1.92139	0.88863	3.79652	2.97274	7.67592	8.65228
0.70000	1.06800	0.14000	1.14096	0.21565	1.39353	0.44076	1.59446	0.61898	1.95532	0.95699	3.92001	3.20141	8.01688	9.31784
0.75000	1.07000	0.15000	1.14520	0.23106	1.40616	0.47224	1.61435	0.66320	1.98925	1.02535	4.04350	3.43008	8.35784	9.98340
0.80000	1.07200	0.16000	1.14944	0.24646	1.41879	0.50372	1.63425	0.70741	2.02319	1.09370	4.16699	3.65876	8.69881	10.64897
0.85000	1.07400	0.17000	1.15368	0.26187	1.43143	0.53520	1.65414	0.75162	2.05712	1.16206	4.29048	3.88743	9.03977	11.31453
0.90000	1.07600	0.18000	1.15792	0.27727	1.44406	0.56669	1.67404	0.79584	2.09106	1.23042	4.41397	4.11610	9.38073	11.98009
0.95000	1.07800	0.19000	1.16216	0.29267	1.45669	0.59817	1.69393	0.84005	2.12499	1.29877	4.53747	4.34477	9.72169	12.64565
1.00000	1.08000	0.20000	1.16640	0.30808	1.46933	0.62965	1.71382	0.88426	2.15892	1.36713	4.66096	4.57344	10.06266	13.31121
1.05000	1.08200	0.21000	1.17064	0.32348	1.48196	0.66113	1.73372	0.92847	2.19286	1.43549	4.78445	4.80212	10.40362	13.97677
1.10000	1.08400	0.22000	1.17488	0.33889	1.49460	0.69262	1.75361	0.97269	2.22679	1.50384	4.90794	5.03079	10.74458	14.64233
1.15000	1.08600	0.23000	1.17912	0.35429	1.50723	0.72410	1.77351	1.01690	2.26073	1.57220	5.03143	5.25946	11.08555	15.30789
1.20000	1.08800	0.24000	1.18336	0.36969	1.51986	0.75558	1.79340	1.06111	2.29466	1.64056	5.15492	5.48813	11.42651	15.97345
1.25000	1.09000	0.25000	1.18760	0.38510	1.53250	0.78707	1.81330	1.10533	2.32860	1.70891	5.27842	5.71681	11.76747	16.63901
1.30000	1.09200	0.26000	1.19184	0.40050	1.54513	0.81855	1.83319	1.14954	2.36253	1.77727	5.40191	5.94548	12.10843	17.30457
1.35000	1.09400	0.27000	1.19608	0.41591	1.55776	0.85003	1.85309	1.19375	2.39646	1.84563	5.52540	6.17415	12.44940	17.97013
1.40000	1.09600	0.28000	1.20032	0.43131	1.57040	0.88151	1.87298	1.23797	2.43040	1.91398	5.64889	6.40282	12.79036	18.63569
1.45000	1.09800	0.29000	1.20456	0.44671	1.58303	0.91300	1.89288	1.28218	2.46433	1.98234	5.77238	6.63150	13.13132	19.30125
1.50000	1.10000	0.30000	1.20880	0.46212	1.59567	0.94448	1.91277	1.32639	2.49827	2.05069	5.89587	6.86017	13.47229	19.96681

Table II
Means and standard deviations for various equity allocations under the rebalancing (RB) strategy
Horizon n = 1, 2, 5, 7, 10, 20, 30 years

Equity	n = 1		n = 2		n= 5		n = 7		n = 10		n = 20		n = 30	
Weight	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev	Mean	StdDev
0.00000	1.04000	0.00000	1.08160	0.00000	1.21665	0.00000	1.31593	0.00000	1.48024	0.00000	2.19112	0.00000	3.24340	0.0000
0.05000	1.04200	0.01000	1.08576	0.01474	1.22840	0.02636	1.33375	0.03387	1.50896	0.04580	2.27695	0.09777	3.43583	0.1807
0.10000	1.04400	0.02000	1.08994	0.02953	1.24023	0.05315	1.35177	0.06855	1.53817	0.09326	2.36597	0.20305	3.63928	0.3829
0.15000	1.04600	0.03000	1.09412	0.04439	1.25216	0.08037	1.37000	0.10409	1.56789	0.14247	2.45829	0.31655	3.85434	0.6091
0.20000	1.04800	0.04000	1.09830	0.05931	1.26417	0.10805	1.38845	0.14052	1.59813	0.19352	2.55403	0.43899	4.08168	0.8624
0.25000	1.05000	0.05000	1.10250	0.07429	1.27628	0.13621	1.40710	0.17788	1.62889	0.24654	2.65330	0.57118	4.32194	1.1460
0.30000	1.05200	0.06000	1.10670	0.08934	1.28848	0.16486	1.42597	0.21623	1.66019	0.30163	2.75623	0.71401	4.57585	1.4638
0.35000	1.05400	0.07000	1.11092	0.10446	1.30078	0.19403	1.44505	0.25560	1.69202	0.35891	2.86294	0.86843	4.84416	1.8199
0.40000	1.05600	0.08000	1.11514	0.11964	1.31317	0.22373	1.46436	0.29605	1.72440	0.41849	2.97357	1.03548	5.12764	2.2192
0.45000	1.05800	0.09000	1.11936	0.13490	1.32565	0.25399	1.48388	0.33762	1.75734	0.48051	3.08826	1.21632	5.42713	2.6669
0.50000	1.06000	0.10000	1.12360	0.15024	1.33823	0.28482	1.50363	0.38036	1.79085	0.54511	3.20714	1.41217	5.74349	3.1693
0.55000	1.06200	0.11000	1.12784	0.16565	1.35090	0.31625	1.52360	0.42432	1.82493	0.61241	3.33035	1.62442	6.07765	3.7333
0.60000	1.06400	0.12000	1.13210	0.18114	1.36367	0.34830	1.54380	0.46955	1.85959	0.68257	3.45806	1.85455	6.43056	4.3666
0.65000	1.06600	0.13000	1.13636	0.19671	1.37653	0.38099	1.56423	0.51612	1.89484	0.75575	3.59041	2.10419	6.80325	5.0783
0.70000	1.06800	0.14000	1.14062	0.21236	1.38949	0.41434	1.58489	0.56407	1.93069	0.83210	3.72756	2.37515	7.19677	5.8784
0.75000	1.07000	0.15000	1.14490	0.22809	1.40255	0.44838	1.60578	0.61346	1.96715	0.91180	3.86968	2.66939	7.61226	6.7785
0.80000	1.07200	0.16000	1.14918	0.24391	1.41571	0.48312	1.62691	0.66436	2.00423	0.99503	4.01694	2.98907	8.05088	7.7917
0.85000	1.07400	0.17000	1.15348	0.25982	1.42896	0.51860	1.64828	0.71682	2.04194	1.08197	4.16952	3.33657	8.51390	8.9332
0.90000	1.07600	0.18000	1.15778	0.27581	1.44232	0.55483	1.66988	0.77091	2.08028	1.17283	4.32758	3.71450	9.00260	10.2201
0.95000	1.07800	0.19000	1.16208	0.29190	1.45577	0.59184	1.69173	0.82670	2.11928	1.26781	4.49133	4.12574	9.51838	11.6719
1.00000	1.08000	0.20000	1.16640	0.30808	1.46933	0.62965	1.71382	0.88426	2.15892	1.36713	4.66096	4.57344	10.06266	13.3112
1.05000	1.08200	0.21000	1.17072	0.32435	1.48298	0.66830	1.73616	0.94366	2.19924	1.47103	4.83666	5.06111	10.63697	15.1637
1.10000	1.08400	0.22000	1.17506	0.34072	1.49674	0.70780	1.75875	1.00496	2.24023	1.57974	5.01864	5.59256	11.24290	17.2591
1.15000	1.08600	0.23000	1.17940	0.35718	1.51060	0.74818	1.78159	1.06825	2.28191	1.69353	5.20711	6.17204	11.88214	19.6311
1.20000	1.08800	0.24000	1.18374	0.37374	1.52456	0.78947	1.80469	1.13361	2.32428	1.81266	5.40229	6.80420	12.55645	22.3187
1.25000	1.09000	0.25000	1.18810	0.39041	1.53862	0.83170	1.82804	1.20112	2.36736	1.93742	5.60441	7.49421	13.26768	25.3668
1.30000	1.09200	0.26000	1.19246	0.40717	1.55279	0.87490	1.85165	1.27086	2.41116	2.06812	5.81370	8.24775	14.01778	28.8270
1.35000	1.09400	0.27000	1.19684	0.42404	1.56706	0.91908	1.87552	1.34292	2.45569	2.20505	6.03040	9.07108	14.80879	32.7585
1.40000	1.09600	0.28000	1.20122	0.44102	1.58144	0.96430	1.89965	1.41740	2.50095	2.34857	6.25477	9.97114	15.64288	37.2298
1.45000	1.09800	0.29000	1.20560	0.45810	1.59592	1.01056	1.92405	1.49438	2.54697	2.49901	6.48704	10.95559	16.52229	42.3199
1.50000	1.10000	0.30000	1.21000	0.47529	1.61051	1.05792	1.94872	1.57396	2.59374	2.65676	6.72750	12.03289	17.44940	48.1198

Table III
Sharpe Ratio as a function of investment horizon, multi-period strategy, and equity weight

Panel A. Multi-period strategy: buy and hold				
Equity Weight, α	$SR_{(1)}$	$SR_{(10)}$	$SR_{(20)}$	$SR_{(30)}$
0.25	0.2000	0.4964	0.5400	0.5123
0.50	0.2000	0.4964	0.5400	0.5123
0.75	0.2000	0.4964	0.5400	0.5123
1.00	0.2000	0.4964	0.5400	0.5123
1.25	0.2000	0.4964	0.5400	0.5123
1.50	0.2000	0.4964	0.5400	0.5123
Panel B: Multi-period strategy: rebalancing				
Equity Weight, α	$SR_{(1)}$	$SR_{(10)}$	$SR_{(20)}$	$SR_{(30)}$
0.25	0.2000	0.6029	0.8092	0.9411
0.50	0.2000	0.5698	0.7195	0.7888
0.75	0.2000	0.5340	0.6288	0.6445
1.00	0.2000	0.4964	0.5400	0.5123
1.25	0.2000	0.4579	0.4555	0.3952
1.50	0.2000	0.4191	0.3770	0.2952

Note: We define the n-period Sharpe Ratio as $SR_{(n)} \equiv \dfrac{E\left(R_{p(n)}\right) - \left[\left(1 + R_{f(1)}\right)^n - 1\right]}{\sigma_{p(n)}}$

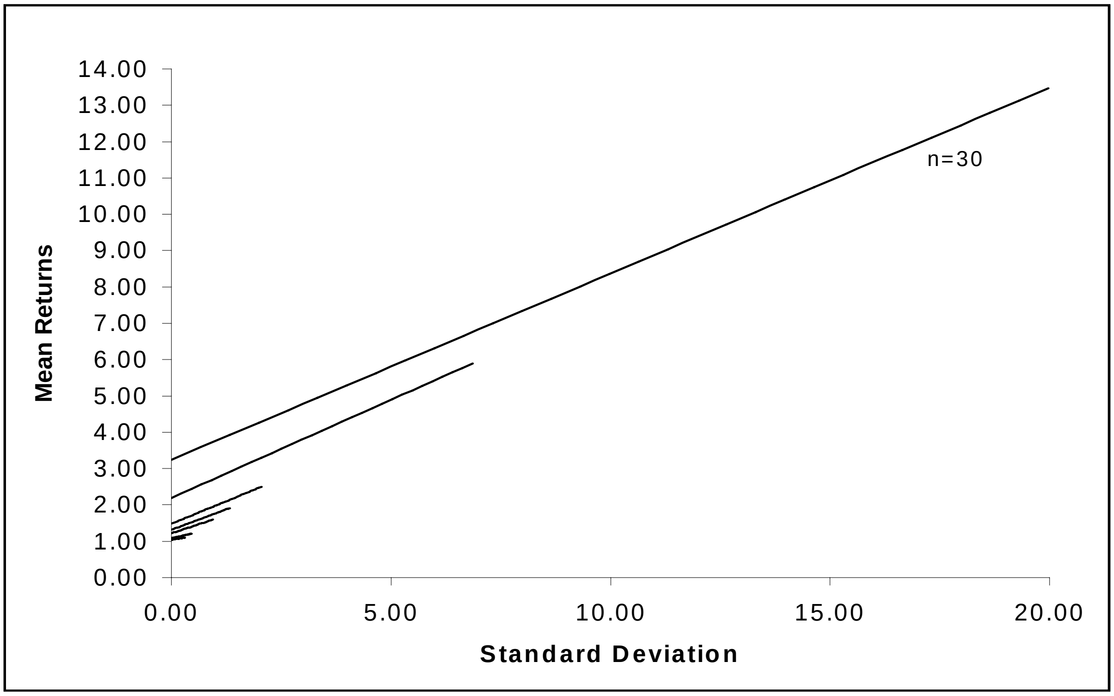


Figure 1.
Mean-standard deviation capital market lines for seven horizons (n = 1, 2, 5, 7, 10, 20, 30 years)
BH strategy

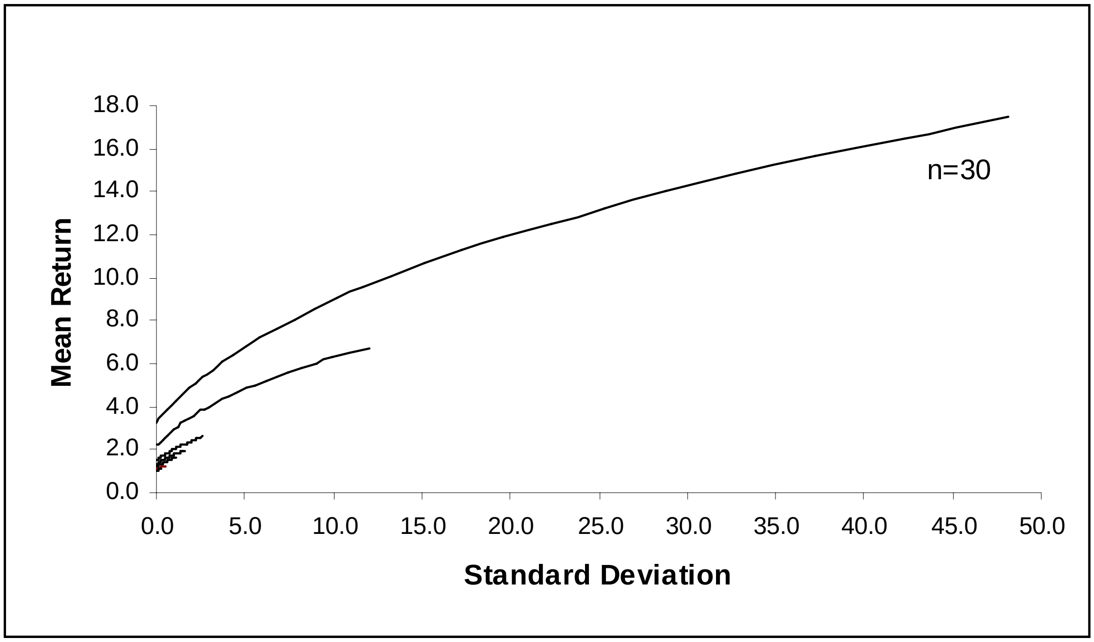


Figure 2.
Mean-standard deviation capital market curves for seven horizons
(n = 1, 2, 5, 7, 10, 20, 30 years) RB strategy

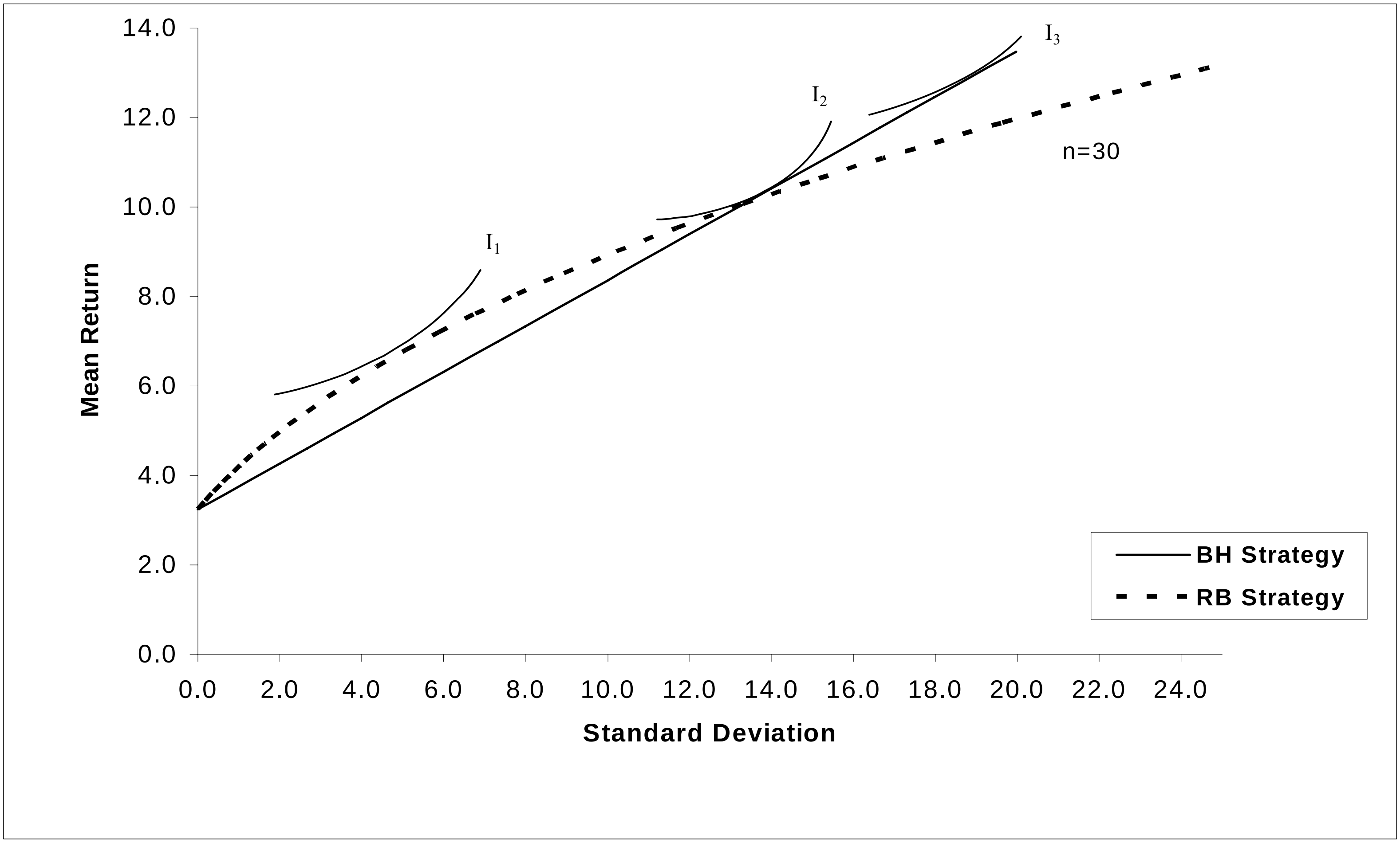


Figure 3.

Mean-standard deviation efficient frontiers under buy and hold and rebalancing strategies; horizon n=30 years

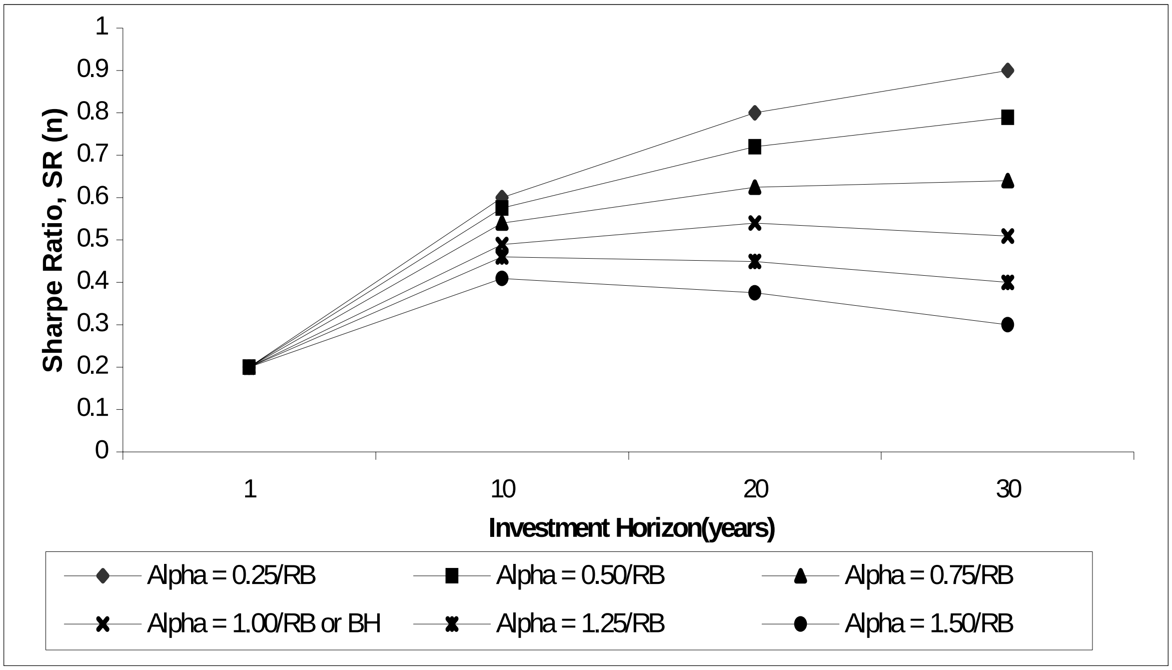
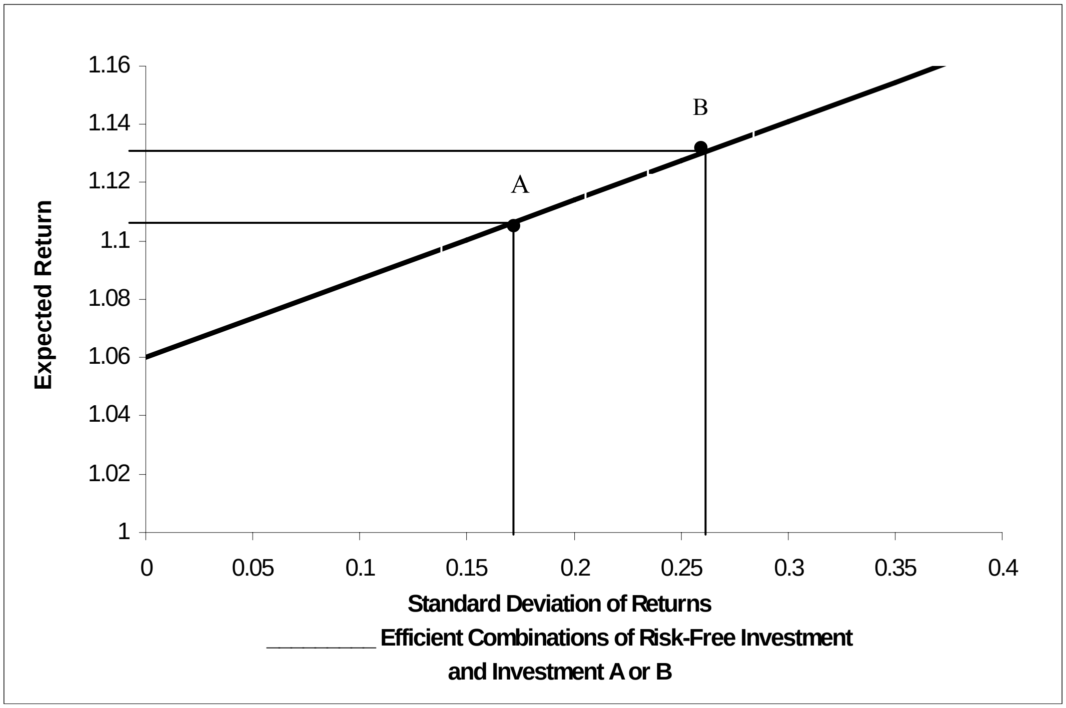


Figure 4.
Sharpe Ratio as a function of equity weight (or risk) α, and investment horizon, n



Note: Point A represents the low-risk investment; Point B represents the high-risk investment.

Figure 5
Risk-return characteristics of two investments: one-year horizon

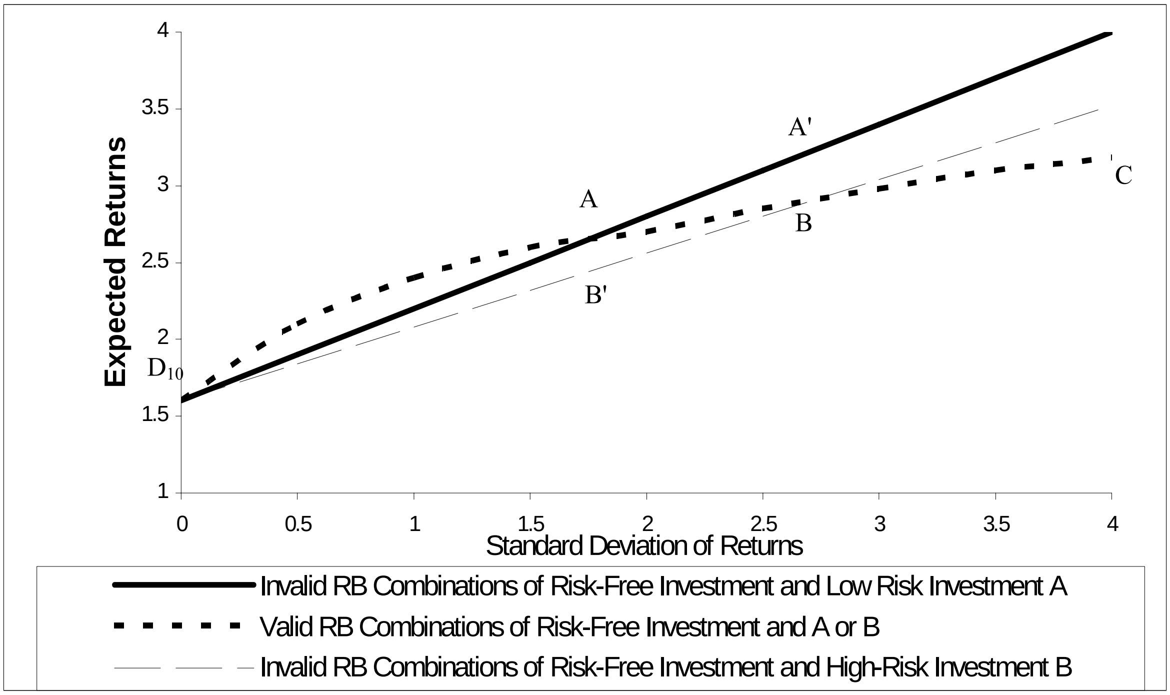


Figure 6
Risk-Return Characteristics of Two Investments: 10-Year Horizon

Appendix A
The Derivation of the BH and RB Formulas

The Buy and Hold Formulas

Under the BH strategy the formula for the n-period portfolio rate of return is:

$$R_p(n) = \alpha\left[(1+R_{m,1})(1+R_{m,2})\ldots(1+R_{m,n})-1\right] + (1-\alpha)\left[(1+R_{f,1})(1+R_{f,2})\ldots(1+R_{f,n})-1\right]$$

$$= \alpha\left[\prod_{i=1}^{n}(1+R_{m,i})-1\right] + (1-\alpha)\left[\prod_{i=1}(1+R_{f,i})-1\right] \qquad (1')$$

where: $R_{m,i}$ (i=1, …, n) is the ith one-period rate of return on the stock index
$R_{f,i}$ (i=1, …, n) is the ith one-period interest rate on the riskless asset

The n-period expected portfolio return is:

$$E(R_p(n)) = \alpha E\left[\prod_{i=1}^{n}(1+R_{m,i})-1\right] + (1-\alpha)E\left[\prod_{i=1}^{n}(1+R_{f,i})-1\right]$$

$$= \alpha\left[\prod_{i=1}^{n}E(1+R_{m,i})-1\right] + (1-\alpha)\left[\prod_{i=1}^{n}E(1+R_{f.i})-1\right] \text{ (invoking independence)}$$

$$= \alpha\left[\prod_{i=1}^{n}((1+E(R_{m,i}))-1\right] + (1-\alpha)\left[\prod_{i=1}^{n}(1+E(R_{f,i}))-1\right]$$

$$= \alpha\left[(1+E(R_{m,i}))^n - 1\right] + (1-\alpha)\left[(1+R_{f,i})^n - 1\right] \text{ (invoking stationarity)}$$

$$= \alpha\left[(1+E(R_m(1)))^n - 1\right] + (1-\alpha)\left[(1+R_f(1))^n - 1\right] \qquad (2')$$

Since $E(R_{m,i}) \equiv E(R_{m,1}) = E(R_m(1))$ and $E(R_{f,i}) = R_{f,i} \equiv R_{f,1}$. Formula (2') is identical to formula (4) contained in the body of the article.

To obtain the formula for the n-period variance, $\sigma_p^2(n)$, we rely on the following expression first derived by Tobin (1965). If $(1+R_{m,i})$ is the one-period return, or wealth relative, then assuming independence over time, the n-period variance is:

$$\sigma_m^2(n) = \prod_{i=1}^{n}\left[(1+E(R_{m,i}))^2 + \sigma_{m,i}^2\right] - \prod_{i=1}^{n}(1+E(R_{m,i}))^2 \qquad (3')$$

where $\sigma_{m,i}^2$ is the ith one-period return variance. With the additional assumption of stationarity, the above expression simplifies to

$$\sigma_m^2(n) = \left[(1+E(R_m(1)))^2 + \sigma_m^2(1)\right]^n - \left[1+E(R_m(1))\right]^{2n} \qquad (4')$$

since $\sigma_{m,1}^2 = \sigma_m^2(1)$. Therefore, the n-period portfolio variance $\sigma_p^2(n)$ is:

$$\sigma_p^2(n) = \alpha^2\sigma_m^2(n)$$
$$= \alpha^2\left\{\left[(1+E(R_m(1)))^2 + \sigma_m^2(1)\right]^2 + \left[1+E(R_m(1))\right]^{2n}\right\} \qquad (5')$$

Which is identical to formula (5) contained in the body of the article.

The Rebalancing Formulas

Under the RB strategy, the n-period portfolio return is

$$1 + R_p(n) = \prod_{i=1}^{n}\left(1 + R_{p,i}\right)$$
$$= \prod_{i=1}^{n}\left(1 + \alpha R_{m,i} + (1-\alpha)R_{f,i}\right)$$

Under the assumption of return independence over time, the expected n-period portfolio return is

$$E\left(1 + R_p(n)\right) = 1 + E\left(R_p(n)\right) = \prod_{i=1}^{n} E\left(1 + R_{p,i}\right)$$
$$= \prod_{i=1}^{n}\left(1 + E\left(R_{p,i}\right)\right)$$
$$= \prod_{i=1}^{n}\left(1 + \alpha\ E\left(R_{m,i}\right) + (1-\alpha)E\left(R_{f,i}\right)\right)$$

With the additional assumption of stationarity of the single-period distribution, from the above expression we obtain the following formula for the n-period expected rate of return:

$$E\left(R_p(n)\right) = \left[1 + \alpha E\left(R_m(1)\right) + (1-\alpha)R_f(1)\right]^n - 1 \qquad (6')$$

Which is identical to formula (6) contained in the body of the article.

To derive the associated n-period portfolio variance $\sigma_p^2(n)$, we employ Tobin's (1965) variance formula, assuming independence and stationarity, and obtain

$$\sigma_p^2(n) = \left[\left(1 + E\left(R_p(1)\right)\right)^2 + \sigma_p^2(1)\right]^n - \left[1 + E\left(R_p(1)\right)\right]^{2n}$$

which is formally identical to (4') with a change in subscript and

$$\sigma_p^2(n) = \left[(1 + \alpha E(R_m(1)) + (1-\alpha)R_f(1))^2 + \alpha^2\sigma_m^2(1)\right]^n - \left[1 + \alpha E(R_m(1)) + (1-\alpha)R_f(1)\right]^{2n} \qquad (7')$$

Which is identical to the formula (7) contained in the body of the article.